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ABSTRACT

Let G and $H \subset G$ be two real semisimple groups defined over $\mathbf{Q}$. Assume that H is the group of points fixed by an involution of G. Let $\pi \subset L^2(H\backslash G)$ be an irreducible representation of G and let $f \in \pi$ be a K-finite function. Let Γ be an arithmetic subgroup of G. The Poincaré series $P_f(g) = \sum_{H\cap\Gamma\backslash\Gamma} f(\gamma g)$ is an automorphic form on $\Gamma\backslash G$. We show that P_f is cuspidal in some cases, when $H \cap \Gamma\backslash H$ is compact.

Introduction

Let G be a real simple group defined over $\mathbf{Q}$ and Γ an arithmetic subgroup. Then $\Gamma\backslash G$ has a finite volume. Let P be a maximal parabolic subgroup in G and let N be its unipotent radical. The subgroup P is called cuspidal if $\Gamma \cap N\backslash N$ is compact. Let $L_0^2(\Gamma\backslash G)$ be the space of square integrable functions f on $\Gamma\backslash G$ such that

$$\int_{\Gamma\cap N\backslash N} f(ng)dn = 0$$

for all cuspidal parabolic subgroups of G. It is well-known that $L_0^2(\Gamma\backslash G)$, the space of cusp forms, decomposes as a sum of irreducible representations of G.

Let τ be an involution on G and H the group of points fixed by τ. Put $X = H\backslash G$. Let $L_d^2(X)$ be the maximal submodule of $L^2(X)$ such that it decomposes as a sum of irreducible representations of G. The description of $L_d^2(X)$ is known and it is due to Flensted-Jensen, Oshima and Matsuki.

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Let P be a maximal parabolic subgroup of G such that $H \cap N = \{1\}$. Put $M_1 = H \cap P$. There is a Langlands decomposition $P = MAN$ such that $M_1 \subset M$. Let $f \in L^2(X)$. Consider

$$f^N(g) = \int_N f(ng)dn.$$

Obviously, f^N might not be defined for every f but we are going to ignore this problem for a moment. Note that f^N is a function on $M_1N\backslash G$. Since the group A acts from the left on $L^2(M_1N\backslash G)$, the spectrum of G on $L^2(M_1N\backslash G)$ is continuous and therefore the intertwining map $f \mapsto f^N$ should vanish on the discrete spectrum $L^2_d(X)$.

Next, suppose that H is **Q**-compact. Then $\Gamma \cap H\backslash H$ is compact. In that case the following implication is a simple consequence of the Hilbert theorem 90:

$$(*) \qquad \Gamma \cap N\backslash N \text{ is compact } \Rightarrow H \cap N = \{1\}$$

Let $f \in L^2_d(X)$. Consider the Poincaré series (disregarding the question of convergence):

$$P_f(g) = \sum_{H\cap\Gamma\backslash\Gamma} f(\gamma g).$$

In view of the statement $(*)$ the cuspidality of P_f follows at once from the vanishing of f^N.

In this paper we study the case $G = SO_0(n+1,1)$ and $H = SO_0(n,1)$. In the first section we give a construction of a part of $L^2_d(X)$ following Flensted-Jensen [F]. In the second section we study some convergence questions related to the study of the map $f \mapsto f^N$. In the third section we show the vanishing of f^N for integrable discrete series. In the fourth section we construct examples of cusp forms. Unfortunately, by Hasse–Minkowski theorem, an arithmetic subgroup Γ, such that $\Gamma\backslash G$ is not compact and such that $H \cap \Gamma\backslash\Gamma$ is compact can be chosen only for $n = 2$ and 3. In the case $n = 3$, the cusp forms of $SO(4,1)$ obtained are non-tempered.

A classical way to construct automorphic forms is via Θ-lift. Since the spectrum of $SO_0(n+1,1)$ on $L^2(X)$ can be described as the Θ-lift from $SL(2)$ (see [H]), it is natural to conjecture that the cusp forms constructed in this paper are the Θ-lift of automorphic forms on $SL(2)$. In fact, Piatetski-Shapiro [P] has shown that the Θ-lift from $SL(2)$ gives cusp forms only when $n \leq 3$, and this is compatible with our results.

Finally, the following remark is due. The map $f \mapsto f^N$ has been used by several people, most notably by Harish-Chandra and Gelfand and his collaborators to study Plancherel formula for symmetric spaces (see [G]).

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NOTATION: *For f and g, two positive functions, $f \asymp g$ means that $c < \frac{f}{g} < \frac{1}{c}$ for some constant $c > 0$.*

1. Constructions of discrete series

Let $\mathbf{R}$ be the field of real numbers and $q = -x_0^2 + x_1^2 + \cdots + x_{n+1}^2$ a quadratic form. Let $G = SO_0(q) = SO_0(n+1,1)$. The Cartan involution θ on G is given by the conjugation by the diagonal matrix $\mathrm{diag}(-1,1,\ldots,1)$. Let τ be the involution given by the conjugation by the diagonal matrix $\mathrm{diag}(1,\ldots,1,-1)$. Let K and H be the groups of points fixed by the involutions θ and τ. Obviously,

$$K = SO(x_1^2 + \cdots + x_n^2) \quad \text{and } H = SO_0(-x_0^2 + x_1^2 + \cdots + x_n^2).$$

Let G_0 be the group of points fixed by the involution $\theta\tau$. Let $\mathbf{g}$ denote the Lie algebra of G. We have the following decompositions into $+1$ and -1 eigenspaces for θ and τ:

$$\mathbf{g} = \mathbf{k} + \mathbf{p},$$
$$\mathbf{g} = \mathbf{h} + \mathbf{q}.$$

Clearly, $\mathbf{k}$ and $\mathbf{h}$ are Lie algebras of K and H and $\mathbf{g}_0 = \mathbf{h} \cap \mathbf{k} + \mathbf{q} \cap \mathbf{p}$ is the Lie algebra of G_0. The vector space $\mathbf{q} \cap \mathbf{p}$ is one dimensional. Let $S \in \mathbf{p} \cap \mathbf{q}$ be such that

$$\exp(tS) = \begin{pmatrix} \cosh t & \ldots & \sinh t \\ \vdots & & \vdots \\ \sinh t & \ldots & \cosh t \end{pmatrix}$$

Let $L = H \cap K$. Then $L = SO(x_1^2 + \cdots + x_n^2)$ and $G_0 = L\exp(q \cap p)$. Let ρ be a holomorphic representation of $K_{\mathbf{C}} = SO(n+1,\mathbf{C})$. Obviously, both H and K are subgroups of $K_{\mathbf{C}}$, and ρ induces a representation of K and H.

THEOREM (Flensted-Jensen): *[F] There is a bijection $f^0 \mapsto f$ between the following two spaces of functions on G:*

(1) *The space of real analytic functions f on $H\backslash G$ which transform as ρ under the action of K from the right.*
(2) *The space of real analytic functions f^0 on $K\backslash G$ which transform as ρ under the action of H from the right.*

The bijection is characterized by $f \mid G_0 = f^0 \mid G_0$. Moreover, if f^0 is equivariant under the action of $\mathcal{Z}(\mathbf{g})$, the center of the universal enveloping algebra $\mathcal{U}(\mathbf{g})$, then f is $\mathcal{Z}(\mathbf{g})$-equivariant as well.

Therefore, to construct a representation on $H\backslash G$, we have to construct f^0. First, we recall the construction of principal series representations for H. Let $q_1 = -x_0^2 + x_1^2 + \cdots + x_n^2$. Then $H = SO_0(q_1)$. Let e be a vector such that $q_1(e) = 0$. Let $P = MAN$ be the subgroup of H stabilizing the line $\mathbf{R}e$. Then P is a parabolic subgroup and A could be identified with $\mathbf{R}^+$ as follows: Let $\mathbf{n}$ be the Lie algebra of N. If $X \in \mathbf{n}$ and $a \in A$ then $Ad(a)X = \alpha(a)X$. The map $a \mapsto \alpha(a)$ is the desired identification. It can be checked that $ae = \alpha(a)e$.

The Principal series $I_s, s \in \mathbf{C}$ is given by the following:

$$I_s = \{f : H \to \mathbf{C} \mid f(mang) = \alpha(a)^{s+\frac{n-1}{2}} f(g), man \in MAN\}$$

Let $V(q_1) \subset \mathbf{R}^{n+1}$ be the cone defined by the equation $q_1 = 0$. It is easy to see that $I_{-m-\frac{n-1}{2}}$ can be also realized as

$$I_{-m-\frac{n-1}{2}} = \{f : V(q_1) \to \mathbf{C} \mid f(yx_0, \ldots, yx_n) = y^m f(x_0, \ldots, x_n), y \in \mathbf{R}\}.$$

PROPOSITION: *Let m be a positive integer. Then $I_{-m-\frac{n-1}{2}}$ contains a finite dimensional representation of H.*

Proof: $I_{-m-\frac{n-1}{2}}$ contains F_m, the space of homogeneous polynomials of degree m. ∎

Remark: Note that F_m contains the L fixed vector given by the function x_0^m.

Let P be a parabolic subgroup of G such that $A \subset H$ and $P \cap H = (M \cap H)A(N \cap H)$ is a parabolic subgroup of H. Let J_s denote the principal series representation for G:

$$J_s = \{f : G \to \mathbf{C} \mid f(mang) = \alpha(a)^{s+\frac{n}{2}} f(g), man \in MAN\}.$$

We have a natural H-invariant map $J_s \to I_{s+\frac{1}{2}}$ given by $f \mapsto f\mid_H$. Recall that there is an H-invariant pairing

$$\langle\ ,\ \rangle : I_s \times I_{-s} \to \mathbf{C}$$

given by

$$\langle f_s, f_{-s}\rangle = \int_L f_s(l) f_{-s}(l)dl,\ f_{\pm s} \in I_{\pm s}.$$

Therefore, we have an H invariant pairing

$$\langle\ ,\ \rangle : F_m \times J_{m+\frac{n}{2}-1} \to \mathbf{C}$$

Let $v \in I_{-m-\frac{n-1}{2}}$ and $w \in J_{m+\frac{n}{2}-1}$ be L and K invariant vector respectively. Normalize v and w such that $v(l) = 1, l \in L$ and $w(k) = 1, k \in K$. By the remark, $v \in F_m$. Define f_m^0 by the following formula:

$$f_m^0(g) = \langle v, g^{-1}w\rangle.$$

The function f_m^0 is not zero since $f_m^0(1) = \operatorname{vol} L$. It is clearly $\mathcal{Z}(\mathbf{g})$ equivariant, right H-finite function on $K\backslash G$. Let f_m be the function on $H\backslash G$ corresponding to f_m^0 via Flensted-Jensen duality. To show that f_m generates a discrete series in $L^2(H\backslash G)$ suffices to check that f_m is square integrable. We need the following proposition:

Proposition ([F]): *Let dh and dk be the Haar measures on H and K. We have the following integration formula on G:*

$$\int_G f(g)dg = \int_H \int_{\mathbf{R}} \int_K f(h\exp(tS)k)dk \cosh^n t dt dh.$$

Since K is compact to check the integrability of f_m suffices to show that

$$\int_{\mathbf{R}} |f_m^0(\exp tS)|^2 \cosh^n t dt < \infty.$$

We have that

$$f_m^0(\exp tS) = \int_L w(l\exp(-tS))dl = \operatorname{vol}(L) w(\exp(-tS))$$

since L is a normal subgroup of G_0. To compute $w(\exp(-tS))$ we have to write $\exp(-tS) = mnak$. Let

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{R}^{n+2}$$

and P be the parabolic stabilizing the line $\mathbf{R}e_1$. Let $\| \ \|$ be the norm on $\mathbf{R}^{n+2}$ given by $x_0^2 + x_1^2 + \cdots + x_{n+1}^2$. If $g = mnak$ then $\|g^{-1}e_1\| = \alpha(a)^{-1}\|e_1\|$. Since $\|\exp(tS)e_1\|^2 = \cosh 2t$ it follows that $|f_m^0(\exp tS)| \asymp (\cosh 2t)^{-\frac{1}{2}(m+n-1)} \asymp (\cosh t)^{-(m+n-1)}$. Therefore, we have obtained the following theorem:

THEOREM: *The function f_m generates a discrete series representation in $L^2(H\backslash G)$ if $m > 0$. The function f_m is integrable if $m > 1$.*

2. The constant term

Let $\Omega \in \mathcal{Z}(\mathbf{g})$ be the Casimir operator. Then $\Omega f_m = \lambda_m f_m$ for some real number λ_m. Let $P = MAN$ be a parabolic subgroup of G such that $H \cap N = \{1\}$. Let f be a function on $H\backslash G$. Define f^N, the constant term of f:

$$f^N(g) = \int_N f(ng)dn.$$

The purpose of this section is to prove the following

PROPOSITION: *Let $m > 0$. Then f_m^N is a smooth function and $\Omega f_m^N = \lambda_m f_m^N$.*

The proof consists of several steps. Any $g \in G$ can be written as $g = h\exp(tS)k$. Consider the vector

$$e_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The stabilizer of e_2 in G is precisely H. On the other hand

$$\begin{pmatrix} \cosh t & \dots & \sinh t \\ \vdots & & \vdots \\ \sinh t & \dots & \cosh t \end{pmatrix} e_2 = \begin{pmatrix} \sinh t \\ \vdots \\ \cosh t \end{pmatrix}.$$

Therefore $\|g^{-1}e_2\|^2 = \cosh 2t \asymp \cosh^2 t$. Let C be any compact subset in G. Then

$$|f_m(ng)| \asymp \|g^{-1}n^{-1}e_2\|^{-(m+n-1)} \asymp \|n^{-1}e_2\|^{-(m+n-1)} \quad \text{for all } g \in C.$$

If we can show that $\|n^{-1}e_2\|^{-(m+n-1)}$ is an integrable function on N then $|f_m|^N$ is a continuous function by the Lebesgue dominated convergence theorem. We need the following:

LEMMA: *Let $|\ |$ be any norm on $\mathbf{n}$, the Lie algebra of N. Let $X \in \mathbf{n}$. Then $\|\exp(X)e_2\| \asymp |X|^2$.*

Proof: To study N we choose the form $q = x_1^2 + \cdots + x_n^2 - 2x_0x_{n+1}$. Then $\mathbf{n}$ can be chosen so that an element $X \in \mathbf{n}$ has the form

$$X = \begin{pmatrix} 0 & x_1, \dots, x_n & 0 \\ & & x_1 \\ & & \vdots \\ & & x_n \\ & & 0 \end{pmatrix} \quad \text{and} \quad |X|^2 = x_1^2 + \cdots + x_n^2.$$

Then $X^2 = |X|^2 Y$ where

$$Y = \begin{pmatrix} 0 & \dots & 1 \\ & \ddots & \vdots \\ & & 0 \end{pmatrix}.$$

By direct observation, if e is a vector such that $Ye = 0$, then there exists $X \in \mathbf{n}$ such that $Xe = 0$.

Claim: If e is a vector such that $\mathrm{Stab}_G(e) = H$ and $H \cap N = \{1\}$ then $Ye \neq 0$.

Indeed, if $Ye = 0$, then there would be $X \in \mathbf{n}$ such that $Xe = 0$ as well. Hence $\exp(X)e = e$ and $\exp(X) \in H$ which is a contradiction. We can now finish the proof of the lemma:

$$\|\exp(X)e_2\| = \|e_2 + Xe_2 + \frac{|X|^2}{2}Ye_2\| \asymp |X|^2. \quad \blacksquare$$

Let B be a unit ball in $\mathbf{n}$. If $m > 0$ then

$$\int_{\mathbf{n}-B} |X|^{-(2m+2n-2)} dX < \infty.$$

Therefore $|f_m|^N$ is a continuous function.

Proof of the Proposition: It is well known that there exists a smooth compactly supported function α on G such that $\alpha * f_m = f_m$. Here

$$\alpha * f_m(g) = \int_G \alpha(x) f_m(gx) dx.$$

Since $|f_m|^N$ is continuous and α compactly supported function, it follows from Fubini theorem that

$$f_m^N = (\alpha * f_m)^N = \alpha * f_m^N.$$

Since $X(\alpha * f_m^N) = (L_X \alpha) * f_m^N$, where L_X denotes the differentiation from the left, it follows that f_m^N is a smooth function and $\Omega f_m^N = \lambda_n f_m^N$. The proposition is proved. ∎

3. The vanishing result

Let $P = MAN$ be a parabolic subgroup in G such that $H \cap N = \{1\}$. In this section we prove the following theorem:

THEOREM: *If $m > 1$ then $f_m^N \equiv 0$.*

We need the following proposition.

PROPOSITION: *Let $P = MAN$ be a parabolic subgroup of G such that $H \cap N = \{1\}$ Let o denote the origin $(H1)$ of the space $H \backslash G$. Then*

(1) *oNA is an open set in $H \backslash G$*
(2) *$f_m^N \in C^\infty(NM \backslash G)$ i.e. f_m^N is left M invariant.*

Proof: It is easy to see that $\dim H \backslash G = \dim NA$. Let $A_1 = H \cap NA$. If $A_1 \neq 0$ then $A_1 \cong A$ since $A_1 \cap N = \{1\}$ and A is connected. Let $\mathbf{a}_1$ be the Lie algebra of A_1. We have the triangular decompositions

$$\mathbf{g} = \mathbf{n}_1^- + \mathbf{m}_1 + \mathbf{a}_1 + \mathbf{n}_1 \quad \text{and} \quad \mathbf{h} = \mathbf{n}_1^- \cap \mathbf{h} + \mathbf{m}_1 \cap \mathbf{h} + \mathbf{a}_1 + \mathbf{n}_1 \cap \mathbf{h}.$$

Obviously $\mathbf{n}_1 = \mathbf{n}$. Since $\mathbf{n}_1 \cap \mathbf{h} \neq 0$ we get $\mathbf{n} \cap \mathbf{h} \neq 0$, a contradiction. The first part is proved. To prove the second part, let $M_1 = H \cap P$. Then $\dim M_1 = \dim M$. Since $M_1 \cap NA = \{1\}$ and M is connected we get $M_1 \to P \backslash NA = M$ is an isomorphism. The proposition is proved. ∎

Proof of the theorem: Consider the function $\varphi(a) = f_m^N(a), a \in A$. We claim that $\varphi \equiv 0$.

LEMMA: *Let G be a simple group and $P = MAN$ a parabolic subgroup. Let $\mathbf{g} = \mathbf{n}^- + \mathbf{m} + \mathbf{a} + \mathbf{n}$ be the corresponding decomposition of the Lie algebra of G. There exist $D \in \mathcal{U}(\mathbf{a})$ of second order such that*

$$\Omega - \Omega_M - D \in \mathbf{n}_{\mathbf{C}}\mathcal{U}(\mathbf{g}),$$

where Ω_M is the Casimir operator for M. ∎

Since Ω is G invariant operator we have $\lambda_m f_m^N = \Omega f_m^N = L_\Omega f_m^N$. The function f_m^N is left M and N invariant. Hence $D\varphi = \lambda_m \varphi$. We know that oNA is an open set in $H\backslash G$. The G invariant measure on $H\backslash G$ restricts to NA invariant measure on oNA. Therefore, it is $dnd^x a$. Since f_m is absolutely integrable function on $H\backslash G$ it follows that φ is integrable function on A. On the other hand, φ is a solution of an ordinary differential equation on A and therefore a linear combination of exponential functions. In particular, φ can be integrable only if $\varphi \equiv 0$. The same conclusion can be obtained for $\varphi_g(a) = f_m^N(ag)$ for all g. The theorem is proved.

4. Application to cusp forms on G

Let Γ be a discrete subgroup of G such that $\mathrm{vol}(\Gamma\backslash G) < \infty$ and $\mathrm{vol}(\Gamma \cap H\backslash H) < \infty$. If $f \in C(H\backslash G) \cap L^1(H\backslash G)$ define Poincaré series by

$$P_f(g) = \sum_{\Gamma\cap H\backslash\Gamma} f(\gamma g).$$

PROPOSITION: *Let $f \in C(H\backslash G) \cap L^1(H\backslash G)$. Assume that there exists a smooth compactly supported function α on G such that $\alpha * f = f$. Then the series P_f converges uniformly and absolutely to a smooth function.*

Proof: (Godement) We have that

$$f(g) = \int_G \alpha(g^{-1}x) f(x)dx = \int_{\Gamma\cap H\backslash G} \sum_{\Gamma\cap H} \alpha(g^{-1}\gamma x) f(x)dx.$$

Therefore

$$\sum_{\Gamma\cap H\backslash\Gamma} |f(\gamma g)| \leq \int_{\Gamma\cap H\backslash G} \sum_{\Gamma} |\alpha(g^{-1}\gamma x)||f(x)|dx.$$

Let C_1 be the support of α. Let C be a compact set. If $g \in C$ and $g^{-1}\gamma_1 x, g^{-1}\gamma_2 x \in C_1$ then $g^{-1}\gamma_1\gamma_2^{-1}g = g^{-1}\gamma_1 x(g^{-1}\gamma_2 x)^{-1} \in C_1C_1^{-1}$. Hence $\gamma_1\gamma_2^{-1} \in CC_1C_1^{-1}C^{-1}$. Let $\beta = \sharp\Gamma \cap CC_1C_1^{-1}C^{-1}$. We have

$$\sum_{\Gamma\cap H\backslash\Gamma} |f(\gamma g)| \leq \beta \operatorname{vol}(\Gamma \cap H\backslash H)\|f\|_1 \quad \text{for all } g \in C.$$

Therefore P_f converges absolutely and uniformly. Since $\alpha * f = f$ Fubini theorem implies that $\alpha * P_f = P_f$, hence P_f is a smooth function. ∎

It is not a priori clear that $P_f \neq 0$. We have the following proposition:

PROPOSITION: *Let f be a function on $\Gamma \cap H\backslash\Gamma$ such that $f(1) \neq 0$ and the series $\sum_{\Gamma\cap H\backslash\Gamma} f(\gamma)$ converges absolutely. Then there is a subgroup Γ' of finite index in Γ such that*

$$\sum_{\Gamma'\cap H\backslash\Gamma'} f(\gamma) \neq 0.$$

Proof: Let Γ_i be a sequence of subgroups in Γ such that $[\Gamma : \Gamma_i] < \infty, \Gamma_i \supset \Gamma_{i+1}$ and $\cap\Gamma_i = 1$. By the Lebesgue dominated convergence theorem it follows that

$$\lim_{i\to\infty} \sum_{\Gamma_i\cap H\backslash\Gamma_i} f(\gamma) = f(1).$$

The proposition is proved. ∎

Recall that G was the connected component (in topological sense) of the real points of an algebraic group. To define an arithmetic group Γ in G suffices to find an algebraic group $\mathcal{G}$ over $\mathbf{Q}$ such that $G = \mathcal{G}(\mathbf{R})^0$ and a $\mathbf{Q}$-embedding ρ of $\mathcal{G}$ into GL_r. Then $\Gamma = G \cap \rho^{-1}(GL_r(\mathbf{Z}))$ is arithmetic. Fix $\mathcal{G}$ and $\mathcal{H} \subset \mathcal{G}$ defined over $\mathbf{Q}$ such that $G = \mathcal{G}(\mathbf{R})^0$ and $H = \mathcal{H}(\mathbf{R})^0$.

THEOREM: *If Γ is an arithmetic subgroup of G such that $\Gamma \cap H\backslash H$ is compact then P_{f_m} is a cusp form. Here $m > 1$.*

Proof: A parabolic subgroup $P = MAN$ is said to be cuspidal if $\Gamma \cap N\backslash N$ is compact. Recall that P_{f_m} is a cusp form on G if and only if

$$\int_{\Gamma\cap N\backslash N} P_{f_m}(ng)dn = 0$$

for all cuspidal parabolic. Let $P = MAN$ be a cuspidal parabolic. Let $P = MAN$ be a cuspidal parabolic. We claim that $H \cap N = \{1\}$. Assume not. Since $\Gamma \cap N\backslash N$

is compact it follows that N is defined over $\mathbf{Q}$. Therefore $H \cap N$ is defined over $\mathbf{Q}$ as well. As an algebraic group $H \cap N$ is just a vector space. The Hilbert theorem 90 implies that $\Gamma \cap H \cap N \neq \{1\}$. But this is impossible since $\Gamma \cap H \backslash H$ is compact and therefore $\Gamma \cap H$ contains no nontrivial unipotent elements. For the same reason $\gamma N \gamma^{-1} \cap H = \{1\}$ for all $\gamma \in \Gamma$. Since

$$\int_{\Gamma \cap N \backslash N} P_{f_m}(ng)dn = \sum_{\Gamma \cap H \backslash \Gamma / \Gamma \cap N} f_m^{\gamma N \gamma^{-1}}(g) = 0$$

the theorem is proved. ∎

The first question is the existence of Γ noncocompact in G such that $\Gamma \cap H$ is cocompact in H. Let $\mathcal{G} = SO(q)$ and $\mathcal{H} = SO(q_1)$ where q and q_1 are rational quadratic forms in $n+2$ and $n+1$ variables of $\mathbf{R}$-index 1. So we ask that q_1 be totally anisotropic over $\mathbf{Q}$. Since over p-adics all forms in at least 5 variables are isotropic, it follows from Hasse-Minkowski that $n = 2$ or 3. In those two cases we make the following choices:

$$\begin{array}{lll} n = 2 & q = -3x_0^2 + x_1^2 + x_2^2 + x_3^2 & q_1 = -3x_0^2 + x_1^2 + x_2^2 \\ n = 3 & q = -7x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 & q_1 = -7x_0^2 + x_1^2 + x_2^2 + x_3^2 \end{array}$$

The anisotropicity of $q_1 (n = 3)$ follows from the following classical result [S]-(p.45):

PROPOSITION: *If $r = x_1^2 + x_2^2 + x_3^2$ where x_1, x_2, x_3 are rational then r is a square in $\mathbf{Q}_2$.*

References

[F] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces,* Ann. of Math. **111** (1980), 253–311.

[G] I. M. Gelfand, M. I. Graev and N. Y. Vilenkin, *Generalized Functions 5,* Academic Press, New York, 1968.

[H] R. Howe, *On some results of Strichartz and of Rallis and Schiffman,* J. Funct. Anal. **32** (1979), 297–303.

[P] I. Piatetski-Shapiro, *On the Saito-Kurakawa lifting,* Invent. Math. **71** (1983), 309–338.

[S] J.-P. Serre, *A Course in Arithmetic,* Graduate Texts in Mathematics 7, Springer-Verlag, Berlin, 1973.